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Azimuthal flow associated with inertial wave resonance in a precessing cylinder

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A series of experiments has been carried out on low-viscosity fluid in a right-circular cylinder that rotates rapidly at a constant speed about its axis of symmetry. This axis in turn is made to undergo less rapid precession about a second axis passing through the centroid of the cylinder. The linear inviscid response of the fluid to such forcing can be expressed as a spectrum of inertial wave modes. However, there are several interesting features of the problem that are associated with nonlinear and viscous effects. One such phenomenon is the appearance of an azimuthal flow under conditions that are related to the underlying linear inertial wave behaviour. Results are presented concerning the manner in which this flow depends on the various experimental parameters. Dynamical properties of the circulation following the onset of forcing have also been investigated. The flow at forcing frequencies close to the fundamental inertial wave resonance was found to have a vortex-like structure, and this led to data that suggest that hydrodynamic instabilities may play a part in the observed breakdown to turbulent motion in regimes of strong forcing.

1. Introduction

Inertial waves are a type of fluid motion found only in rotating systems, since it is the Coriolis force that provides the necessary restoring effect to give rise to oscillations. When a rotating body of fluid is subjected to periodic forcing, inertial waves may be excited with angular frequencies ranging from zero up to a maximum of twice the basic rotation rate (Greenspan 1968). Considerable attention has been given to problems in which inertial waves are forced by precession of the axis of rotation. Precessional forcing has the advantage over other forms excitation, such as the oscillating disk (Fultz 1959), that there is no intrusion of obstacles into the flow. Also, there are a number of physical applications where precession is a factor. These include motions in the Earth's liquid core (Vanyo *et al.* 1995), the dynamics of spin-stabilized projectiles (Herbert 1986) and the stability of rotating spacecraft with fluid payloads (Pocha 1987).

The latter problem was the motivation for a previous study by the present author of the dynamics of inertial waves in a rotating and precessing right-circular cylinder (Kobine 1995). In the course of investigating experimentally the limitations of the linear and inviscid approximations that are commonly adopted in analytical and computational treatments of the problem, it was observed that an azimuthal flow

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could arise in addition to the expected flow due to inertial wave motion. The purpose of this paper is to present results from a specific experimental investigation of the properties of the azimuthal flow, and the manner in which they depend on the various experimental parameters.

Busse (1968) showed analytically that a steady fluid motion could arise in a rotating fluid-filled sphere undergoing precession. He based his calculations on the linear boundary layer analysis carried out by Stewartson & Roberts (1963) and Roberts & Stewartson (1965), but extended to include nonlinear terms at finite amplitudes. The main conclusion was that a localized shear layer could be induced in the otherwise stationary interior fluid (as seen from the rotating reference frame) by the nonlinear advection of the boundary-layer velocity field. More recently, experiments by Vanyo *et al.* (1995) have shown that flow is in fact induced throughout the bulk of the fluid. This has been explained by Kerswell (1995) and Hollerbach & Kerswell (1995) in terms of boundary layer eruptions giving rise to obliquely inclined shear layers that propagate into the fluid interior.

A similar treatment to that of Busse (1968) was given by Thompson (1970) to the problem of a tilted rotating cylinder with a free surface, which results in forcing with a very similar character to that of precessional forcing. Thompson's calculations predicted the existence of a series of resonant depths at which a central vortex motion would appear, and this prediction was confirmed experimentally using visualization techniques. A comprehensive experimental study was made by McEwan (1970) of inertial wave phenomena in a cylinder with an inclined rotating lid. Once again, evidence was found for an induced azimuthal flow under certain forcing conditions, this time by the ingenious use of a free-turning wire anemometer inserted into the flow. Vaughn, Oberkampf & Wolfe (1985) have shown numerically that a weak azimuthal flow is induced by precessional forcing in highly viscous fluids. In the regime of low viscosity, however, there is a scarcity of numerical schemes that do not resort to inviscid approximations. The studies carried out by Selmi, Li & Herbert (1992) and Hall, Sedney & Gerber (1992) based on spatial eigenfunction expansions are examples of attempts to include boundary-layer effects at small Ekman numbers. However, the focus is on calculating wall pressures and induced moments, and it remains the case that there are no known data available for explicit azimuthal velocities in this regime.

The existence of a nonlinearly induced azimuthal flow at small Ekman numbers is especially significant in connection with the violent resonant collapse phenomena that are observed under conditions of strong forcing in precessing systems. Such phenomena, in which an initially laminar flow configuration breaks down into highly turbulent motion, represent a considerable challenge from the point of view of the underlying fluid mechanical processes. Manasseh (1992) has shown experimentally that for the case of a spinning and precessing cylinder, there are several qualitatively distinct routes by which the flow may become turbulent. One such route involves a number of partial relaminarizations and secondary collapses following the initial breakdown event. Such behaviour is not accounted for using a classical stability analysis. However, a possible mechanism has been suggested by Gunn & Aldridge (1990) involving a net flow that effectively detunes the system by shifting the value of the resonant frequency. Gunn & Aldridge formulated their analysis in terms of prescribed azimuthal flows in a cylindrical domain, and considered the effects on the frequency and amplitude of the fundamental inertial mode of the system with axisymmetric forcing. The results were used to show that the process of detuning could be supported at least qualitatively by the existence of a net circulation.

Although there have now been several reported studies involving the qualitative

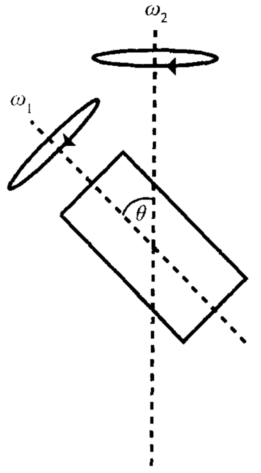

FIGURE 1. Schematic representation of precessional forcing. Cylinder rotates at speed ω_1 about its axis of symmetry. Rotation axis precesses at speed ω_2 in laboratory reference frame.

aspects of circulation phenomena in precessing systems, there appears to be very little quantitative information available. The objective of the present study was to obtain a variety of such data which would be of use in understanding the properties of precessionally driven flows in greater detail, as well as pointing the way for any future analytical and computation investigations of the problem. The flows were investigated experimentally with a laser velocimeter, which provided accurate and instantaneous measurements of the fluid velocity without introducing any disturbances in the flow. To begin with, the appearance of the azimuthal flow was studied with quasi-static variation of the forcing frequency, particularly for frequencies approaching the fundamental inertial wave resonance of the system. The study was then extended to consider the time-dependent evolution of the flow following the onset of forcing close to resonance. Finally, measurements were made of the radial dependence of the azimuthal speed, which in turn allowed some conjectures to be made regarding the hydrodynamic stability of the induced flow.

2. Experimental details

The configuration of the problem under consideration is shown schematically in figure 1. The cylinder rotates about its axis of symmetry, which in turn precesses about an axis passing through the centroid of the cylinder. The angular speed of precession as measured from the inertial frame of reference is ω_2. The angular speed of the cylinder about its axis of symmetry is ω_1 relative to the precessional frame of reference. The dimensionless frequency Ω of the precessional forcing is defined here as

$$\Omega = \frac{\omega_1}{\omega_1 + \omega_2}. \tag{1}$$

The other experimental parameter is the tilt angle θ between the two axes.

The experimental apparatus that was used to produce the above configuration was the same as that described by Kobine (1995), where full details may be obtained. Only the essential elements are summarized here. The rotating cylinder was made from Perspex and was filled completely with distilled water. The cylinder had an internal radius $a = 45.00 \pm 0.02$ mm and an internal height $H = 117.0 \pm 0.1$ mm, giving an aspect ratio $h \equiv H/2a = 1.300 \pm 0.002$. The angular speed ω_1 of the cylinder was stable to within 0.1% at a typical operating speed of 10.0 rad s^{-1}. At this speed, the Ekman number $E = \nu/a^2\omega_1$ is approximately 5×10^{-5}, where ν is the kinematic viscosity of the fluid. Tilting of the cylinder was performed by a stepping motor, with a minimum angular increment of $\Delta\theta = 0.018°$ and a tilt speed of $\mathrm{d}\theta/\mathrm{d}t = 2.88°$ s^{-1}. The cylinder was mounted on a turntable which rotated with an angular speed ω_2 that was stable to within 0.2%. The speed of the turntable was controlled by a microcomputer, with the typical range being 0–3 rad s^{-1}. Initial alignment of the rotation axes of the cylinder and turntable was performed manually with reference to a spirit level, giving a zero setting that was accurate to $\pm 0.1°$.

A purpose-built miniature laser velocimeter was used to measure the azimuthal component of flow velocity at a point inside the cylinder. Full details of this device have been given previously by Kobine (1995). The velocimeter was mounted so as to look down through the transparent upper face of the cylinder, with the lower face being a Perspex mirror which returned the beams through the top face for processing. Two different modes of operation were possible. One involved the velocimeter being attached to the structure that held the rotating cylinder. In this way, the velocimeter tilted with the cylinder but was otherwise stationary in the precessional reference frame rotating with speed ω_2. Alternatively, the velocimeter could be attached directly to the cylinder, in which case measurements were made in the frame in which the cylinder appears stationary. Most measurements were made with the cylinder rotating relative to the velocimeter, since the signal could only be tracked by the processing electronics if there was sufficient flow (at least 10 mm s^{-1}) through the measuring volume. Also, although the velocimeter was not directionally sensitive, the flow due to solid-body rotation provided an effective frequency shift of the Doppler signal to allow the direction of any induced flow to be determined. However, this mode of operation resulted in lower signal-to-noise levels than when the velocimeter rotated with the cylinder. The choice of operational mode will be indicated in each case.

The measuring point of the velocimeter was fixed in the axial direction at a distance of 40 mm from the upper face of the cylinder. The radial position was variable, and is quoted throughout in terms of the dimensionless coordinate $r = r^*/a$. In the case where the cylinder rotated relative to the velocimeter, the azimuthal coordinate ϕ, as measured from the axis about which the cylinder was tilted, was always equal to zero.

3. Development of azimuthal flow with variation of forcing frequency

3.1. *Experimental procedure*

With both the cylinder and the turntable at rest, the axis of symmetry of the cylinder was tilted out to the required angle θ and subsequently held fixed. The cylinder was then set rotating at a constant angular speed ω_1 and left for a time well in excess of the theoretical spin-up time. Once the fluid was known to be in solid-body rotation, the angular speed ω_2 of the turntable was increased slowly from zero to a chosen final value ω_f. The time-dependent dimensionless forcing frequency $\Omega(t)$ in this case is given by

$$\Omega(t) = \frac{\omega_1}{\omega_1 + \omega_2(t)}, \tag{2a}$$

where

$$\omega_2(t) = \begin{cases} 0 & \text{for } t < 0; \\ \omega_f t/T & \text{for } 0 \leqslant t \leqslant T; \\ \omega_f & \text{for } t > T. \end{cases} \tag{2b}$$

The constant angular acceleration ω_f/T of the turntable was always chosen to be sufficiently small that the time dependence of ω_2, and hence of Ω, could be assumed to have no effect on the observed flow behaviour.

The laser velocimeter was mounted on the support that held the rotating cylinder, ensuring that it tilted with the cylinder but was otherwise stationary with respect to the turntable. It was necessary to measure from this frame of reference because of the requirement of sufficient flow through the measuring volume of the velocimeter. This was provided at all times by the solid-body rotation of the fluid, and also allowed the direction of the induced flow to be established relative to the basic rotation. The time series of azimuthal flow speed was recorded during the acceleration phase of the turntable, with time subsequently being converted into dimensionless forcing frequency using (2).

3.2. *Prograde precession*

The fundamental resonance of the system is known to exist in the regime of prograde precession, where the cylinder and the turntable rotate in the same direction. The cylinder rotation speed was set to a constant value of $\omega_1 = 10.0$ rad s^{-1}. The turntable rotation speed was described by (2*b*), with $\omega_f = 3.0$ rad s^{-1} and $T = 1200$ s, giving a constant angular acceleration of 2.5×10^{-3} rad s^{-2}. In this way, the dimensionless forcing frequency was varied from $\Omega = 1$ at the start of the experiment to $\Omega = 0.769$ at the end. The total time corresponded to approximately 1900 rotations of the fluid-filled cylinder in the turntable frame of reference.

Experiments were performed at fixed values of the tilt angle θ ranging from 0.25° to 2.0°, with the measuring point at $r = 0.22$ in each case. Examples of the recorded flow behaviour are shown in figure 2(*a*, *b*). In both cases, the experiment began at $\Omega = 1$ and with the azimuthal flow speed as measured by the velocimeter corresponding to the speed associated with solid-body rotation. Initially, as the turntable speed was increased gradually from zero, the fluid continued to rotate as a solid body. However, with further increase in the turntable speed, and therefore further decrease in Ω, an obvious departure from solid-body rotation was observed. For the case of $\theta = 0.25°$ as shown in figure 2(*a*), the azimuthal flow speed at the measurement point increased smoothly with decrease of Ω to a maximum value at $\Omega = 0.783$ of approximately 10% greater than the initial speed. This excess flow subsequently decreased with further decrease of Ω. At the larger tilt angle of $\theta = 1.0°$, a somewhat different type of behaviour was observed. This is illustrated in figure 2(*b*). The fluid motion in the early stages of the experiment was again simply solid-body rotation. As was the case for $\theta = 0.25°$, the flow was observed to accelerate with further decrease of Ω. However, the forcing amplitude in this case was sufficiently large to cause obvious oscillatory behaviour to be excited in addition to the unidirectional azimuthal flow. Unlike the response at the smaller tilt angle, the azimuthal flow speed continued to increase as the value of Ω was decreased. The variation of the average azimuthal flow speed, as shown in figure 2, was highly repeatable between different experiments at the same tilt angle. However, the exact details of the oscillatory behaviour at the larger tilt angles were different for each experiment.

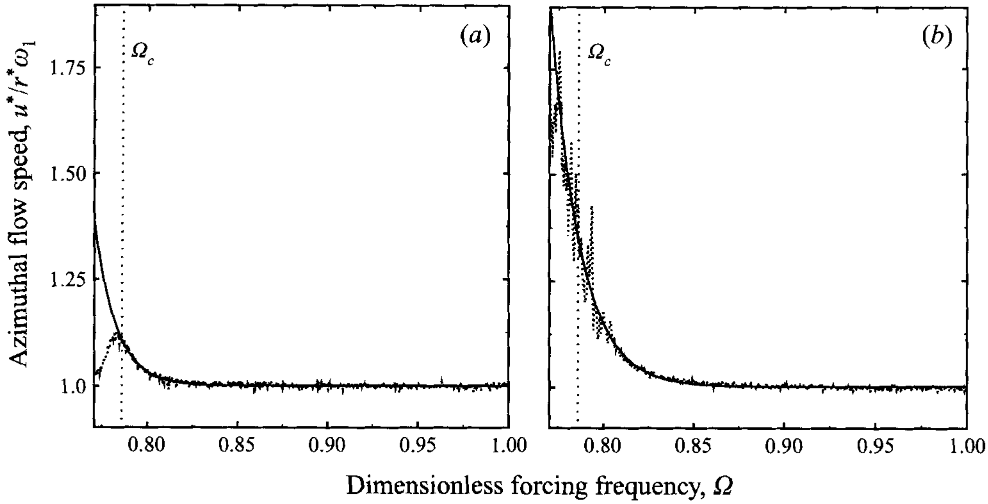

FIGURE 2. Development of mean azimuthal flow with prograde precessional forcing at $r = 0.22$: (*a*) $\theta = 0.25°$; (*b*) $\theta = 1.0°$. Dotted lines are experimental results. Solid lines are least-squares fits of (3) with $\Omega_c = 0.786$ as marked. Time evolution of forcing frequency is from right to left.

3.3. *An empirical model for the mean azimuthal speed*

The qualitative form of the growth phase of the azimuthal component was found to be the same for all values of θ that were investigated. This observation led to the development of an empirical model for the mean azimuthal speed $\bar{u}$ as a function of the dimensionless forcing frequency Ω. The model that was found to be the most successful in describing the experimental data is

$$\bar{u}(\Omega) = 1 + U \exp\big(\alpha(\Omega_c - \Omega)\big), \tag{3}$$

where non-dimensionalization is with respect to the solid-body rotation speed $u_{sbr} = r^*\omega_1$ at the point of measurement. The first term on the right-hand side of (3) corresponds to solid-body rotation, which exists as a matter of course. The second term represents the azimuthal flow that is driven by the precessional forcing. The terms U, α and Ω_c are all taken to be constants for any particular experiment.

The model given by (3) requires some particular value Ω_c of the forcing frequency to be identified. The experimental trace shown in figure 2(*a*) for $\theta = 0.25°$ proves instructive in that respect. The value of the forcing frequency at which the maximum azimuthal flow speed was recorded ($\Omega = 0.783$) is within 0.4% of the resonant frequency of the fundamental inertial mode of the system ($\Omega = 0.786$) as measured experimentally by Kobine (1995). Thus, it was considered justifiable on pragmatic grounds to take $\Omega_c = 0.786$ in the subsequent discussion. The model was fitted to the experimental data using the method of least squares, with U and α as the free parameters of the fit. Only data for $\Omega_c \leqslant \Omega \leqslant 1$ were used in the fitting process because of the different responses at $\Omega < \Omega_c$ as discussed in §3.2. The results for $\theta = 0.25°$ and $1.0°$ are the solid lines in figure 2(*a*, *b*). In both cases, the model accurately describes the mean component of the azimuthal flow for values of the forcing frequency starting at $\Omega = 1$ and decreasing to $\Omega = \Omega_c$. In the case of $\theta = 0.25°$, the model and the experiment diverge for $\Omega < \Omega_c$. However, for $\theta = 1.0°$, the model continues to follow the mean behaviour for values of the forcing frequency less than Ω_c.

Clearly (3) is not a complete model. As mentioned above, it is not accurate for $\Omega < \Omega_c$ at certain values of θ. Also, it predicts a non-zero circulation even when $\Omega = 1$ and hence when there is no precessional forcing. Nevertheless, the excellent agreement with the experimental data for the majority of the range of forcing frequency that was studied gives encouragement that (3) does at least capture the essential behaviour and is therefore worth continuing to investigate.

3.4. *Assessment of contributions to the azimuthal flow*

The linear inviscid theory of inertial waves in a right-circular cylinder allows the fluid response to be decomposed into a spectrum of normal modes (Kelvin 1880). There is a particular velocity field associated with each normal mode which gives rise to a component of flow in the azimuthal direction. For the case of precessional forcing, only modes with azimuthal wavenumber $m = 1$ are excited, and it can be shown that such modes have a spatial structure that is stationary with respect to the precessional frame of reference. However, the conjecture here is that the azimuthal velocity that is observed experimentally consists of an additional contribution that is related to nonlinear and viscous effects occurring at the boundaries. Thus, it is necessary to show that the underlying inertial wave spectrum is not the sole cause of the observed behaviour.

A relationship between the amplitude of the inertial wave response and the forcing amplitude has been given by McEwan (1970) in the case of a rotating cylinder where the fluid is forced by a precessing lid. The assumption was made that a balance between the precessional forcing and viscous dissipation gives rise to a 'steady-state' amplitude q. This leads to a relationship of the form

$$q \propto a\omega\alpha \tag{4}$$

where ω is the cylinder rotation speed and α is the angle of inclination of the precessing lid. The constant of proportionality in (4) was found by McEwan to be of unit order.

Experiments carried out by Kobine (1995) have shown that a linear relationship exists between the azimuthal velocity and the tilt angle in the case of the fundamental inertial mode in a rotating right-circular cylinder. This was found by applying quasi-impulsive forcing to the fluid in solid-body rotation by means of a sudden tilt of the rotation axis. The subsequent azimuthal flow response as measured by the laser velocimeter took the form of an exponentially decaying sinusoid. The maximum amplitude of this transient response was measured over a range of tilt angles, and was found to obey direct proportionality for angles up to approximately 2.5°. A slight deviation from the linear trend for larger angles up to 4° was attributed to the increasing time required to execute the tilt.

Returning to the present configuration of precessional forcing, the amplitude coefficient U in (3) represents the magnitude of the mean azimuthal flow speed in excess of solid-body rotation at the forcing frequency $\Omega = \Omega_c$. The values of U that were obtained from the fitting process described in §3.3 were noted for each value of θ at which experiments were performed. The results are plotted in figure 3(*a*). The solid line drawn through the experimental data is the result of a least-squares fit of a third-order polynomial in θ, where $U \sim O(\theta)$ to leading order (shown as the dotted line).

For very small amplitudes ($\theta \ll 1°$), the azimuthal component does appear to be following the linear dependence on θ that is expected from the inertial mode alone. However, the measured values very quickly depart from this linear trend with

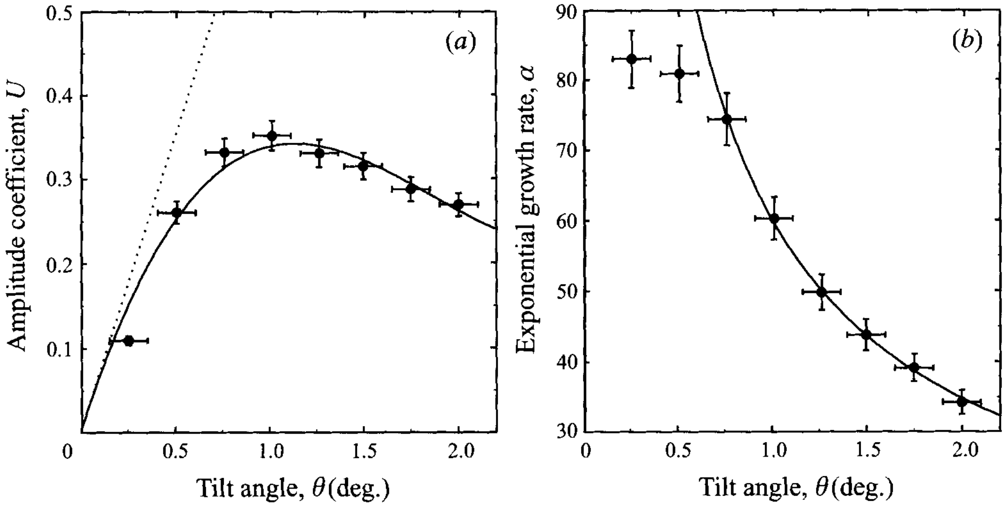

FIGURE 3. Variation of terms in empirical model of mean flow with tilt angle θ: (*a*) amplitude U, solid line is least-squares fit of third-order polynomial, dotted line is expected azimuthal speed from inertial mode only; (*b*) growth rate α, solid line is least-squares fit of inverse power law.

increasing θ, indicating that there is some significant additional contribution to the azimuthal flow. Thus it is concluded that the linear inviscid response cannot be the sole cause of the observed azimuthal flow, and that other factors related to nonlinear boundary-layer effects must also be considered.

3.5. *Variation of growth rate with tilt angle*

The measured variation with tilt angle of the exponential growth rate α, obtained from least-squares fits of (3), is plotted in figure 3(*b*). The value of α decreases monotonically with increase of θ. Indeed, it was found that the variation of α with θ for $\theta \geqslant 0.75°$ followed a power law of the form $\alpha \sim \theta^{-c}$, with a value of $c = 0.785$ giving the best agreement with the experimental data. The departure from this power-law behaviour for $\theta < 0.75°$ is assumed to be due to the non-zero angular acceleration of the turntable during the experiments. Larger values of α correspond to larger values of $|\mathrm{d}\bar{u}/\mathrm{d}\Omega|$, but the rate at which the fluid can respond to such changes is obviously limited by the spin-up time scale. Such scaling behaviour would be a potential source of comparison with any future theoretical or numerical treatments of this particular problem.

3.6. *Retrograde precession*

In order to test the wider applicability of the empirical model developed in §3.3, experiments similar to those described in §3.2 were performed for the case of *retrograde* precession, where the cylinder and the turntable rotate in opposite directions. This was achieved in practice by making the cylinder rotate in the opposite direction to the case of prograde precession, while keeping the turntable direction the same. The cylinder speed was fixed at a value of $\omega_1 = -10.0$ rad s^{-1}, while the turntable rotation speed was again described by (2*b*) but now with $\omega_f = 1.2$ rad s^{-1} and $T = 1200$ s. This resulted in a variation of the dimensionless forcing frequency from $\Omega = 1$ at the start of the experiment to $\Omega = 1.136$ at the end. Once again, the experiments were repeated at different values of the tilt angle θ.

Examples of the fluid response in terms of the azimuthal velocity component at

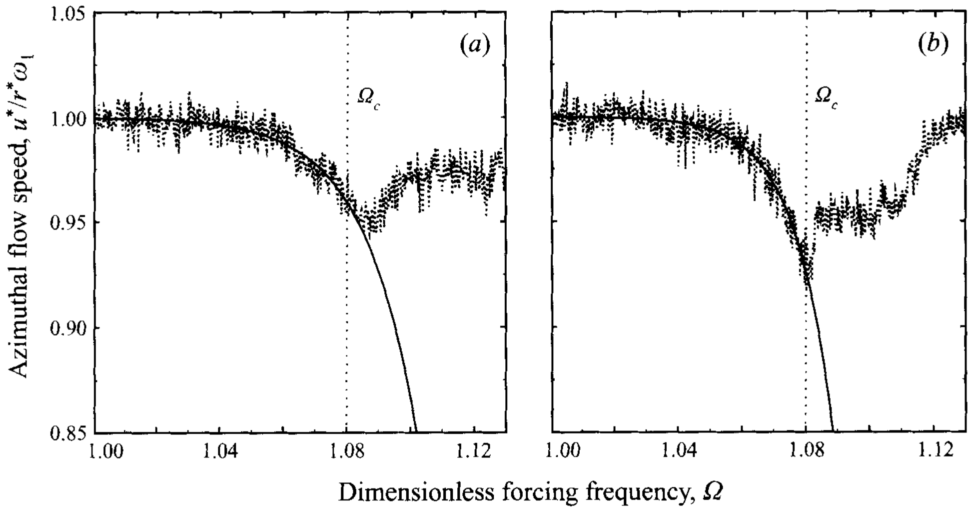

FIGURE 4. Development of mean azimuthal flow with retrograde precessional forcing at $r = 0.24$: (*a*) $\theta = 2.5°$; (*b*) $\theta = 5.5°$. Dotted lines are experimental results. Solid lines are least-squares fits of (3) with $\Omega_c = 1.08$ as marked. Time evolution of forcing frequency is from left to right.

$r = 0.24$ are shown in figures 4(*a*) and 4(*b*) for $\theta = 2.5°$ and 5.5° respectively. As was the case for prograde forcing, the fluid is initially in solid-body rotation. Then, as Ω is increased, there is a departure from the trivial rotation, corresponding to an induced azimuthal flow. In this case, the flow is in the *opposite* direction to the rotation of the cylinder and has a magnitude that is significantly reduced compared to the magnitude of the flow that was driven by the prograde forcing. For each value of θ at which experiments were performed, the azimuthal flow grew to some maximum size before decaying with further increase of Ω. The noise on the signal is an unavoidable feature of the velocimeter (see Kobine 1995), and appears larger here than in figure 2 only because of the magnification of the velocity scale.

The growth phase of the flow is again described successfully by the empirical rule given in (3). The solid lines in figure 4 are the results of least-squares fits of (3) to the experimental data for $1 \leqslant \Omega \leqslant \Omega_c$, with $\Omega_c = 1.080$. This value corresponds to the value of Ω at which the maximum departure from solid-body rotation was measured in the case of $\theta = 5.5°$ in figure 4(*b*). Manasseh (1992) has calculated the resonant frequencies of some of the lower-order modes for the case of a precessing cylinder with aspect ratio $h = 1.333$, where the azimuthal wavenumber m is required to be equal to 1 because of the nature of the forcing. In general, the dimensionless frequency of an inertial mode with axial wavenumber k, radial wavenumber l and azimuthal wavenumber $m = 1$ in a cylinder with aspect ratio h is given by

$$\Omega = 2\left(1 + \left(\frac{2\lambda_{kl1}h}{k\pi}\right)^2\right)^{-1/2}. \tag{5a}$$

The term λ_{kl1} in (5*a*) is the lth root of

$$\lambda J_1'(\lambda) + \left(1 + \left(\frac{2\lambda h}{k\pi}\right)^2\right)^{1/2} J_1(\lambda) = 0, \tag{5b}$$

where $J_1(\lambda)$ is the Bessel function of the first kind of order 1.

Equation (5*b*) was solved numerically for the present case of aspect ratio $h = 1.300$. In particular, it was found that there is a mode with resonant frequency $\Omega = 1.07755$ which has axial wavenumber $k = 3$, radial wavenumber $l = 3$ and azimuthal wavenumber $m = 1$. The value of $\Omega_c = 1.080$ which was used in fitting (3) to the experimental data is within 0.2% of this inviscid theoretical value. Johnson (1967) has shown that viscous effects result in a first-order correction of $O(E^{1/2})$ to the frequencies of inviscid inertial modes in a cylinder. This is equal to a shift of approximately 0.7% in the present case, which is greater than the above discrepancy. Thus it would appear that the mean flow that was driven by retrograde precessional forcing is associated with an inertial wave resonance in the same way that the circulation observed for prograde forcing was found to be connected with the fundamental inertial mode.

4. Azimuthal flow following sudden onset of precessional forcing

The experiments discussed in §3 were designed to establish the manner in which the azimuthal flow develops with quasi-static variation of the forcing frequency for fixed values of the tilt angle. The objective of the experiments described in this section was to measure the time-dependent growth of the flow when precessional forcing was started suddenly.

4.1. *Immediate flow response*

Each individual experiment began with the rotation axes of the cylinder and the turntable being coincident. The cylinder and turntable rotated at constant speeds of $\omega_1 = 10.0$ rad s^{-1} and $\omega_2 = 2.79$ rad s^{-1} respectively. These two values give a dimensionless forcing frequency of $\Omega \equiv \omega_1/(\omega_1 + \omega_2) = 0.782$, which is within 0.5% of the frequency of the fundamental inertial mode ($\Omega = 0.786$). The velocimeter was mounted so as to tilt with the cylinder but was otherwise stationary in the turntable reference frame. The system was left for a time well in excess of the theoretical spin-up time, after which the cylinder was tilted out to some angle θ. Recording of the azimuthal flow speed was started at the same moment that the tilt was initiated. The experiment was repeated for various values of the tilt angle.

The dotted line in figure 5 is an example of the fluid response immediately following the onset of resonant precessional forcing. The recording was made at $r = 0.24$, with tilt angle $\theta = 2.0°$. The azimuthal flow speed at time $t = 0$ corresponded to solid-body rotation. As soon as forcing began, the flow speed increased, accompanied by the onset of an oscillation with growing amplitude. As with the behaviour found in §3, the variation of the average speed was highly repeatable between different runs, but the phase of the oscillation was found to vary. The nature of the growing oscillation has been investigated previously (Kobine 1995), and it is the behaviour of the mean component that is of interest here.

It was found that the growth of the mean azimuthal speed over approximately three cylinder revolutions following the onset of forcing could be modelled empirically by a power law of the form

$$\bar{u}(t) = 1 + \kappa t^{\beta}. \qquad (6)$$

The solid line in figure 5 is the result of a least-squares fit of (6) to the experimental data. This process was repeated for similar time series that were obtained at tilt angles ranging from $\theta = 1.0°$ to $2.75°$. The value of β that was obtained from the fitting process was noted in each case, and the results are plotted in figure 6(*a*).

The power-law index β showed no systematic variation with tilt angle θ outside the

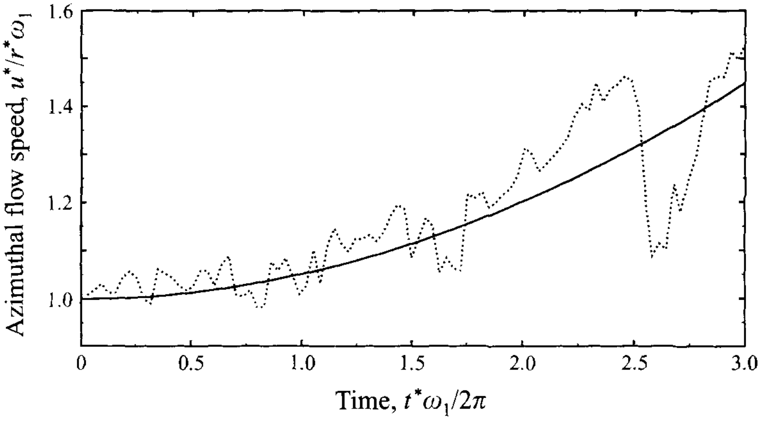

FIGURE 5. Time-dependent growth of azimuthal flow immediately following onset of precessional forcing. $\Omega = 0.782$, $\theta = 2.0°$, $r = 0.24$. Dotted line is experimental result. Solid line is least-squares fit of $u = 1 + \kappa t^{\beta}$.

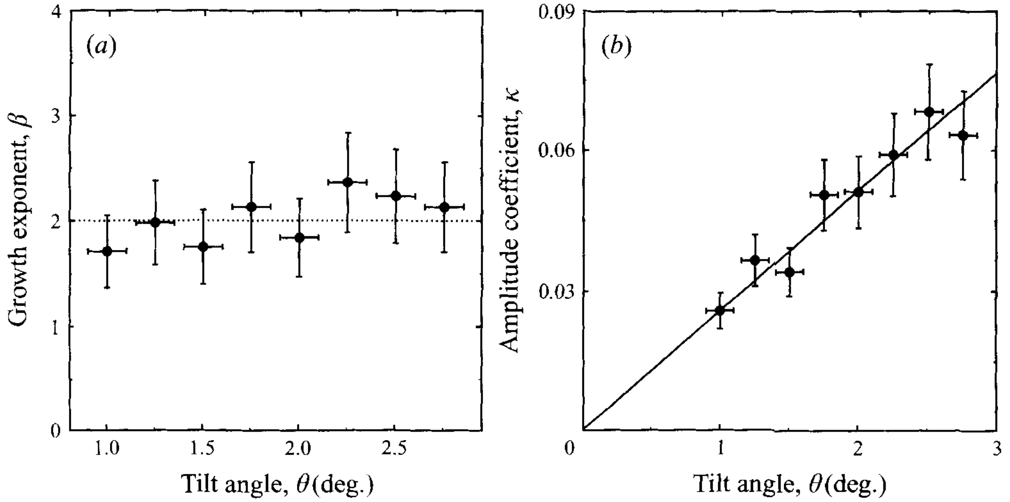

FIGURE 6. Variation of terms in empirical model of time-dependent growth of mean flow with tilt angle θ: (*a*) growth exponent β; (*b*) amplitude coefficient κ for the case $\beta = 2$, solid line is least-squares fit of $\kappa = k\theta$.

limits of experimental error. Indeed, the data support a constant value of $\beta = 2$ over all values of θ that were investigated. The general model given by (6) was therefore taken specifically as

$$\bar{u}(t) = 1 + \kappa t^2, \tag{7}$$

and was fitted once again to each of the experimental data sets in order to establish the manner in which the amplitude coefficient κ varies with tilt angle θ. The values of κ which were obtained in this way are plotted in figure 6(*b*). The growth amplitude does show a systematic variation with tilt angle. The solid line in figure 6(*b*) is a fit to the experimental data of the linear relationship $\kappa = k\theta$, with $k = 0.0257$ deg^{-1}.

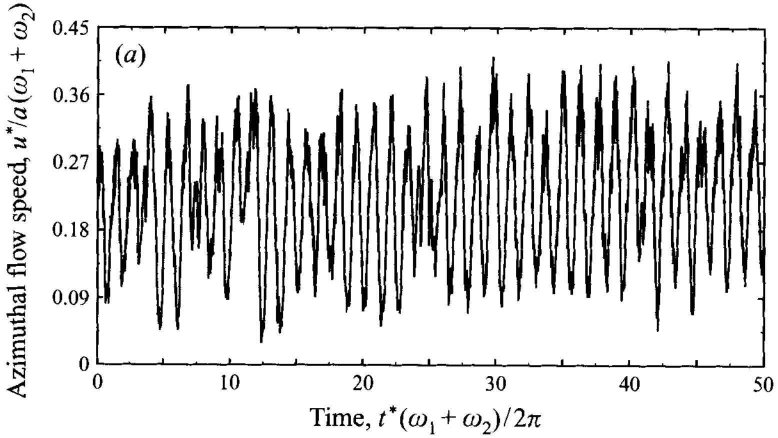

FIGURE 7. Time series of azimuthal flow speed following initial growth phase. $\Omega = 0.782$, $\theta = 2.0°$, $r = 0.19$. Recording made from reference frame rotating with the cylinder.

4.2. *Subsequent flow response*

A second set of experiments was performed to determine the subsequent development of the average azimuthal flow following the initial $O(t^2)$ growth phase. The angular speed of the cylinder was 10.0 rad s^{-1} as before, but several different turntable speeds were used, resulting in values of Ω in the range 0.72–0.80. The tilt angle in each case was $\theta = 2.0°$ and the measuring position was $r = 0.19$. This time the miniature velocimeter was attached directly to the rotating cylinder. This configuration not only allowed measurements to be made in the frame of reference in which the cylinder was stationary, but also improved the signal-to-noise ratio of the velocimeter signal. It was possible to make measurements in this way because sufficient flow had been established through the measuring volume due to the initial acceleration described in §4.1.

A typical time series of the azimuthal velocity following the initial growth phase is shown in figure 7 for $\Omega = 0.782$. The feature that is immediately obvious is that the mean component of the flow response no longer continues to grow as $\bar{u} \sim t^2$. Instead, the rate of growth slows down and eventually the mean component saturates at some apparently constant value. In order to quantify such observations, time series such as the one shown in figure 7 were processed in such a way that successive maxima and minima of the inertial oscillations were located. Taking the mean value of pairs of these measurements was found to give a reasonable time series of the mean azimuthal flow speed.

The results of the above process applied to the time series in figure 7 are shown in figure 8. It was found that the variation of the mean azimuthal speed with time is approximately linear between dimensionless times of $t = 4$ and 8. A least-squares fit of

$$\bar{u} = \gamma t + b \tag{8}$$

was carried out over this range, and the result is the solid line in figure 8. The experimental data for $t > 8$ diverge from this linear trend as the flow speed attains some limiting value on average.

The experiment was repeated at various values of the forcing frequency Ω in the range 0.72–0.80, with both the tilt angle and the measuring position remaining constant. In each case, a time series of mean azimuthal speed was extracted from the

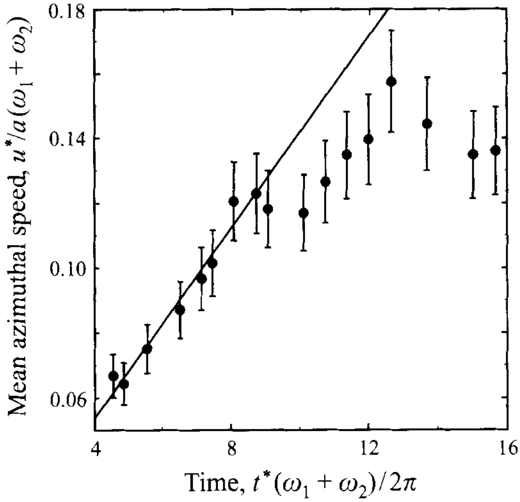

FIGURE 8. Time series of mean azimuthal speed obtained from recording shown in figure 7. Data support linear growth for time t between approximately 4 and 8 units.

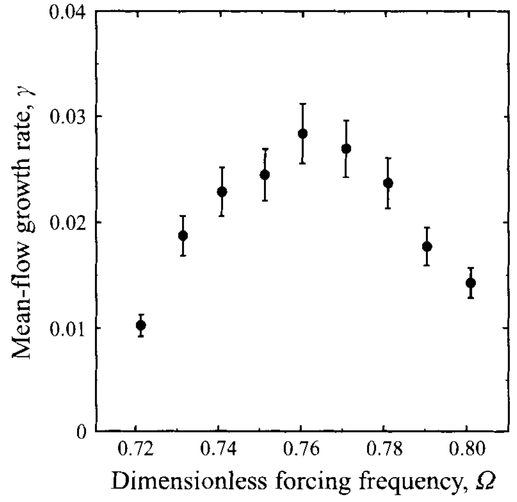

FIGURE 9. Variation of linear mean-flow growth rate γ with forcing frequency. $\theta = 2.0°$, $r = 0.19$.

oscillatory response in the manner described above, and the linear phase was fitted with (8) to obtain a value for the dimensionless mean-flow growth rate γ. The results of this process are plotted in figure 9. The experimentally determined value of γ was found to vary systematically with Ω, with a maximum value at $\Omega \approx 0.76$ in the range that was studied.

It is instructive to compare the behaviour of the mean component as discussed here with that of the oscillatory behaviour, which was investigated experimentally by Kobine (1995). In that study, it was found that the amplitude of inertial oscillations following the sudden onset of precessional forcing also grows linearly with time until approximately the same stage that the mean flow variation is found to depart from its linear trend (figure 8). Measurements made of the dimensionless amplitude growth rate over the range $\Omega = 0.72$ to 0.80 for $\theta = 2.0°$ showed no systematic variation of this quantity with Ω. This raises the question as to why the amplitude growth rate should be insensitive to the forcing frequency, while the mean-flow growth rate shows a clear dependence. Also, having established in §3 that the existence of the mean circulation is related in some sense to the underlying spectrum of inertial waves, it is perhaps surprising that the maximum mean-flow growth rate does not coincide with the resonant frequency of the fundamental inertial mode. Further investigations are required before such questions can be answered.

4.3. *Saturated flow response*

After the linear growth phase discussed in §4.2, the mean azimuthal speed saturates at some constant value. It proved instructive to consider the dependence of this time-averaged constant speed on the forcing amplitude given by the tilt angle θ. These data were obtained from data sets that were recorded in previous experiments carried out by Kobine (1995) to investigate the spectral characteristics of flow in the fully developed nonlinear regime.

The cylinder speed was $\omega_1 = 10.67$ rad s^{-1} and the turntable speed was $\omega_2 = 2.99$ rad s^{-1}, giving a value for the dimensionless forcing frequency of $\Omega = 0.781$. Initially the two axes of rotation were coincident, and the fluid was allowed to attain solid-body rotation. The cylinder was then tilted out to the required angle θ, which was in the range 3–5°. At these tilt angles and at the chosen value of Ω, the flow was always found to undergo breakdown to disordered motion as described by Manasseh (1992). The cylinder was left for at least 120 revolutions, as seen from the turntable reference frame, in order to ensure that all breakdown events and any transient behaviour had occurred. Recordings of the azimuthal speed, lasting for approximately 500 revolutions, were then made with the laser velocimeter attached directly to the rotating cylinder and therefore measuring with respect to tank-fixed coordinates. The radial measuring position was $r = 0.39$ in each case.

An example of a typical time series that was recorded in this way is shown in figure 10(a). If the response was due simply to the velocimeter seeing the spatial structure of the fundamental inertial mode, which is stationary in the precessing frame, then the time series would correspond to a rectified sine wave (since the velocimeter is directionally insensitive). However, the non-zero minima of the oscillation are further evidence of the additional azimuthal flow that is being caused by the forcing. The mean value $\bar{u}^*$ of the minima in the time series was calculated in each case in order to obtain a quantitative measure of the strength of the nonlinearly-induced azimuthal flow. Non-dimensionalization was with respect to the solid-body rotation speed $r^*\omega_1$ at the measuring point. The results are plotted in figure 10(b), where the variation with θ can be seen. There is clearly a systematic variation of $\bar{u}$ with θ, with the strength of the induced flow increasing with tilt angle.

5. Radial structure of the azimuthal flow

So far, the properties of the azimuthal flow have been investigated at fixed measuring points with variation of two experimental parameters, namely the forcing frequency and tilt angle. Results are presented in this section from a series of repeated experiments in which the radial coordinate of the measuring position was varied. In this way, it proved possible to obtain information about the variation of the azimuthal flow speed in the radial direction. This allows some conjectures to be made about the stability of the flow in relation to observations of the breakdown of inertial waves to highly disordered motion.

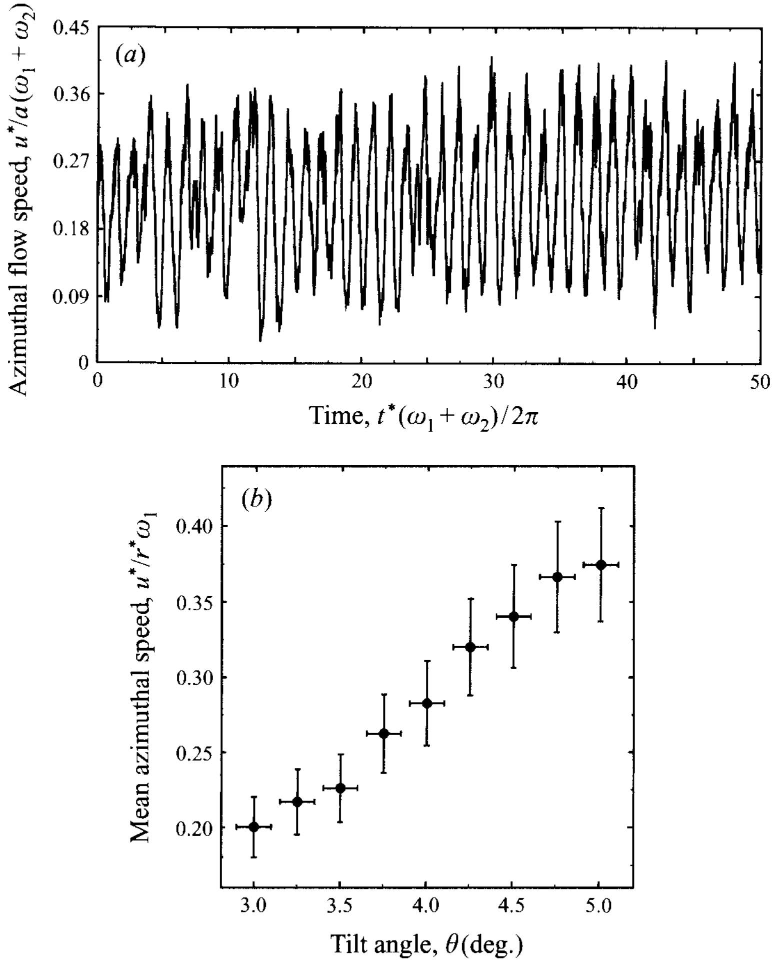

FIGURE 10. Saturated response to precessional forcing measured from cylinder reference frame at $\Omega = 0.781$: (*a*) time series of azimuthal speed for $\theta = 4.0°$; (*b*) variation of induced mean speed with tilt angle. Cylinder speed $\omega_1 = 10.67$ rad s^{-1}, turntable speed $\omega_2 = 2.99$ rad s^{-1}, measuring position $r = 0.39$.

5.1. *Experimental procedure*

Individual experiments were the same in general as those described in §3. The tilt angle and cylinder rotation speed were preset to constant values, and the turntable rotation speed was increased gradually from zero to some final value in the regime of prograde precession. The difference here is that the experiments were repeated several times with the measuring point of the velocimeter at different radial positions within the flow.

In each case, the tilt angle was set to the value $\theta = 1.0°$ at the beginning of the experiment. The cylinder angular speed was fixed at $\omega_1 = 10.0$ rad s^{-1}, and the fluid was left to spin-up to solid-body rotation. Once this was achieved, the turntable speed was increased linearly from zero in accordance with (2*b*). The final angular speed ω_f was chosen to be 3.0 rad s^{-1}, with a ramp time T of 1200 s. The fluid response was measured from the turntable frame of reference and was recorded during the time in which the turntable was accelerating.

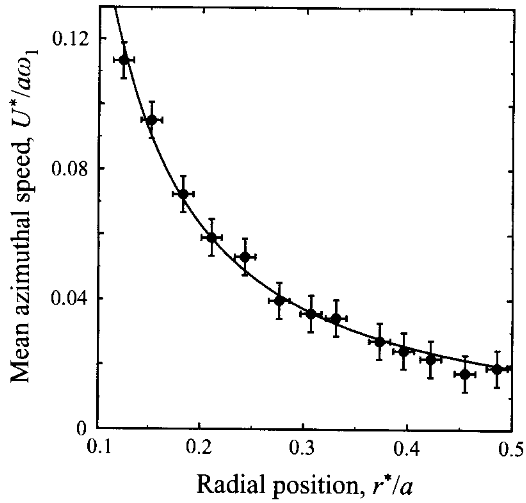

FIGURE 11. Variation of mean azimuthal speed U at $\Omega = 0.786$ with radial position. $\theta = 1.0°$. Solid line is least-squares fit of $U = Ar^{-c}$, with $c = 1.30$.

The resulting variation of azimuthal flow speed with forcing frequency was similar in each case to the plot shown in figure 2(*b*). The growth of the mean-flow speed as Ω decreased from $\Omega = 1$ was modelled as before by (3), with a least-squares fitting procedure giving the value of the amplitude coefficient U. This value was noted for each experimental run, with the dimensionless radial coordinate $r = r^*/a$ of the measuring position being in the range $r = 0.1$ to 0.5.

5.2. *Variation of U with r*

The term U in (3) represents the magnitude of the average azimuthal flow speed at forcing frequency $\Omega = \Omega_c$. For the purpose of this section of the investigation, values of U were rescaled with respect to the azimuthal speed $a\omega_1$ of the cylindrical wall in order to compare speeds measured at different radial positions. The measured variation of U with dimensionless radial coordinate r is shown in figure 11 for the case of $\Omega_c = 0.786$ and $\theta = 1.0°$. The results indicate that the strength of the azimuthal circulation at $\Omega = 0.786$ decreases away from the rotation axis of the cylinder. This is in contrast with fluid moving in solid-body rotation, where the azimuthal flow speed increases linearly from zero with distance from the axis of rotation. It proved possible to describe the variation of U with r accurately in terms of a power law of the form

$$U(r) = Ar^{-c}, \tag{9}$$

with A, r and c all greater than zero. The solid line in figure 11 is the result of a least-squares fit of (9) to the experimental data, giving a value of the exponent $c = 1.30 \pm 0.04$, where the error is derived from the fitting process.

Circulations in which the azimuthal speed increases towards the centre are normally associated with vortex flows. In particular, for the case of a straight line vortex of infinite length in an unbounded fluid, the resulting two-dimensional circulation can be shown analytically to have the form $u \sim r^{-1}$ (Batchelor 1967). In the case of the present investigation, the fluid is bounded in both its radial and axial extent, so such an idealization is far from accurate. Nevertheless, it would seem reasonable to infer from the above experimental evidence that the induced azimuthal flow is related to some form of finite line vortex lying along the axis of symmetry of the cylinder.

The fact that the best fit of (9) to the experimental data is obtained with $c \approx 1$ is considered significant in this respect.

There are also the experimental observations made by Manasseh (1992, 1994) using flow visualization of a weak circulation (referred to by Manasseh as the *anomaly drift*) in a rotating cylinder prior to the commencement of precessional forcing at the fundamental inertial resonance. It was noted by Manasseh (1992) that the initial zero setting of the tilt angle was subject to an uncertainty of $\pm 0.1°$. It now seems likely that Manasseh's 'anomaly drift' is in fact the precursor of the much stronger azimuthal flow that has been revealed by the present experiments, and that the former flow was being driven by forcing with very small amplitude resulting from the tilt angle not being exactly equal to zero.

5.3. *Variation of power-law index with forcing frequency*

The analysis carried out in §5.2 of the azimuthal flow field was for the particular case of $\Omega = 0.786$, which is the experimentally determined frequency of the fundamental resonance for this particular system. However, it was also possible to investigate the structure of the azimuthal flow at other values of the forcing frequency. This was done by using the least-squares fits of (3) that were carried out on the experimental data sets obtained at the different radial measuring positions. For any particular value of Ω, the corresponding mean azimuthal flow speed could be recovered. The only restriction was that the value of Ω should lie in the range $\Omega_c < \Omega < 1$, since it was over this range that the fits were carried out. This method of generating data concerning the flow had the advantage over using the raw experimental data that noise and oscillations on the time series were filtered out by the fitting process.

It was found that the radial distributions of azimuthal speed that were obtained in the range $\Omega = 0.786$ to 0.816 could all be modelled accurately by the power law given by (9). As was the case for the data obtained at $\Omega = 0.786$, (9) was fitted to each data set in order to obtain a value for the power-law index c. The results are plotted in figure 12, with individual error bars having been derived from the fitting process. Clearly there is a high degree of systematic variation of c with Ω. The data plotted in figure 12 support a simple linear relationship over the range that was investigated.

5.4. *Application of Rayleigh's stability criterion*

It is well known that a rotating body of fluid can experience instability as a result of an adverse distribution of angular momentum. In the absence of viscosity, the stability of a two-dimensional circular flow with azimuthal speed $U = U(r)$ is determined by *Rayleigh's criterion* (see Chandrasekhar 1961, for example). If the quantity

$$\Phi(r) = \frac{\mathrm{d}}{\mathrm{d}r}(rU)^2 \tag{10}$$

is defined as the discriminant for stability, then Rayleigh's criterion is that the velocity field $U(r)$ is stable only if

$$\Phi(r) > 0 \tag{11}$$

everywhere in the interval of r over which the flow extends. The velocity field of interest here has the analytical form given by (9). Substituting (9) for U in (10) gives an expression for the discriminant Φ as

$$\Phi(r) = 2A^2(1-c)r^{1-2c}. \tag{12}$$

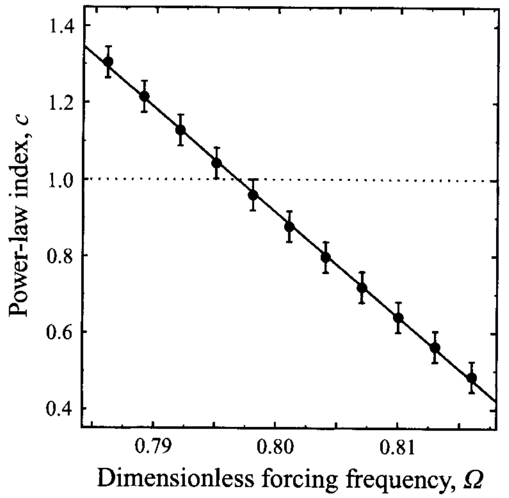

FIGURE 12. Variation of radial power-law index c with forcing frequency. Solid line is least-squares fit of linear relationship. Flows with values of c greater than unity may be hydrodynamically unstable.

Since r is required to be positive, the condition for stability given by (11) reduces to

$$c < 1 \tag{13}$$

in the present case.

Referring to the results plotted in figure 12, it is clear that the condition given by (13) is only satisfied for values of the dimensionless forcing frequency above a certain value, which is found to be $\Omega = 0.79674$ from the straight line through the data. The fact that the flow field continues to display a power-law variation in the radial direction for values of Ω less than this critical value indicates that the flow does not become hydrodynamically unstable at that point. The action of viscosity results in the stabilization of what, in the inviscid case, ought to be an unstable flow. Nevertheless, the trend of the power-law index c with decreasing Ω gives reasonable grounds to suppose that the mean circulation would be prone to hydrodynamic instability with further reduction of the forcing frequency or at larger values of the forcing amplitude, which is determined by the tilt angle.

6. Conclusions

A variety of quantitative experimental results has been presented that illustrates the properties of the azimuthal flow that arises in a rotating right-circular cylinder undergoing precessional motion. The main contribution to the observed flow is due to nonlinear and viscous effects, which are neglected from most analytical and numerical studies of the problem at small Ekman numbers. Such higher-order viscous effects have been known to occur in principle since the analytical work of Busse (1968) on precessing spheroids, but the present study has shown specifically how the azimuthal component of flow depends on the accessible experimental parameters. Both quasi-static and dynamical properties have been investigated, and a number of relatively simple empirical models have been found to describe the experimental observations.

The most significant findings of the investigation are considered to be that the flow induced at forcing frequencies close to the fundamental inertial wave resonance appears to have a vortex-like structure, and that the distribution of angular momentum in the flow violates Rayleigh's stability criterion for forcing frequencies below a critical value. Such information is pertinent when considering the violent breakdowns to turbulence that are found in a number of precessionally driven rotating flows. On the basis of an analysis in terms of inertial modes only, it might be thought that such breakdown phenomena are the result of 'wave breaking' due to nonlinear effects at finite amplitudes. However, it would now appear that a hydrodynamic instability could also be involved. Such a possibility has in fact been suggested by Gunn & Aldridge (1990). If a mean circulation exists and is on the brink of centrifugal instability, then the growth of inertial oscillations could result in the triggering of full-scale collapse comprising both hydrodynamic instability and inertial wave breaking.

It is hoped that the prospects for further analytical work on precessionally forced flows have been improved by the present experimental investigation. The fact that simple behaviour has been observed, such as the exponential variation of mean azimuthal speed with forcing frequency, gives reason to believe that analytical extensions of the problem are feasible. From the point of view of further experiments, an immediate priority should be to obtain data at various azimuthal and axial positions so that the question of whether or not there is some form of line vortex lying along the axis of the cylinder can be resolved. A related endeavour would be to investigate the flow behaviour in an annular domain, where a core vortex would no longer be permissible. It is encouraging that Selmi & Herbert (1995) have shown numerically that such a change of topology has a pronounced effect on the destabilizing moments induced by flow in partially filled precessing cylinders.

Finally, in the context of a technological problem such as the stability of rotating spacecraft with liquid payloads, the existence of an induced azimuthal circulation has serious implications for any attempt to model the problem using computational means. The approach in general is to adopt linear and inviscid approximations in order to circumvent mathematical difficulties associated with the full equations of motion. This is sufficient for dealing with the exchange of angular momentum between the solid and liquid mass fractions due to inertial oscillations associated with vehicle precession. However, to neglect the nonlinear and viscous terms is to ignore the phenomenon of induced azimuthal flow, which the present study has shown can be a major element in the response of the contained fluid.

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